J. Fluid Mech. (1982), *vol.* 124, *pp.* 239–259
Printed in Great Britain

Flames as gasdynamic discontinuities

By M. MATALON AND B. J. MATKOWSKY
Department of Engineering Sciences and Applied Mathematics, The Technological Institute, Northwestern University, Evanston, Illinois 60201

(Received 22 March 1982 and in revised form 27 May 1982)

Early treatments of flames as gasdynamic discontinuities in a fluid flow are based on several hypotheses and/or on phenomenological assumptions. The simplest and earliest of such analyses, by Landau and by Darrieus prescribed the flame speed to be constant. Thus, in their analysis they ignored the structure of the flame, i.e. the details of chemical reactions, and transport processes. Employing this model to study the stability of a plane flame, they concluded that plane flames are unconditionally unstable. Yet plane flames are observed in the laboratory. To overcome this difficulty, others have attempted to improve on this model, generally through phenomenological assumptions to replace the assumption of constant velocity. In the present work we take flame structure into account and derive an equation for the propagation of the discontinuity surface for arbitrary flame shapes in general fluid flows. The structure of the flame is considered to consist of a boundary layer in which the chemical reactions occur, located inside another boundary layer in which transport processes dominate. We employ the method of matched asymptotic expansions to obtain an equation for the evolution of the shape and location of the flame front. Matching the boundary-layer solutions to the outer gasdynamic flow, we derive the appropriate jump conditions across the front. We also derive an equation for the vorticity produced in the flame, and briefly discuss the stability of a plane flame, obtaining corrections to the formula of Landau and Darrieus.

1. Introduction

The equations governing flame propagation are extremely complicated. They involve the Navier–Stokes equations for viscous compressible flows, expressing the conservation of mass and momentum, coupled with the transport equations governing heat conduction and diffusion of the chemical species participating in exothermic chemical reaction(s), expressing the conservation of energy and species. The coupling from the fluid equations to the transport equations occurs for example through convection, while the coupling in the other direction is due to the thermal expansion of the gas in which combustion takes place. The equations involve not only the usual fluid-dynamical nonlinearities, but also the exponentially nonlinear term describing the Arrhenius chemical reaction rate. It is therefore not surprising that scientists have resorted to approximate treatments, either by introducing phenomenological models or by considering various limiting cases, thus permitting analysis of the resulting model.

For example, a diffusional thermal model was introduced by Barenblatt, Zeldovich & Istratov (1962). In this model, which is related to the often-employed constant-density approximation, the effect of thermal expansion is considered to be small, so that a decoupling of the fluid equations from the transport equations is effected. Thus, to leading order, the fluid equations in the absence of thermal expansion are solved,

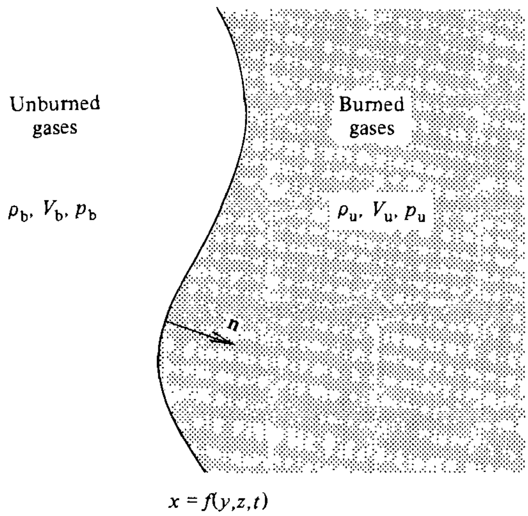

FIGURE 1. Schematic representation of a flame as a gasdynamic discontinuity.

and the resulting velocity and constant-density fields are employed as known coefficients in the transport equations. The effect of the fluid field on the flame is thus taken into account, while the effect of the flame on the gas through which it propagates is ignored. A consistent mathematical derivation by asymptotic methods of a diffusional–thermal-type model from the general equations of combustion was presented by Matkowsky & Sivashinsky (1979). In a sequel, van Harten & Matkowsky (1982) included the effects of a weak coupling. These models have been successfully employed to describe qualitatively various flame phenomena, including cellular flames as well as pulsating and spinning flames.

At the opposite end of the spectrum, there are models that account fully for thermal expansion, but suppress the roles of transport processes. A general description of these models is given in Markstein (1964), as follows. A flame was regarded as a moving density discontinuity surface called the flame front, separating burned from unburned gas (cf. figure 1). The fluid variables were assumed to suffer jump discontinuities across the front, corresponding to statements of conservation of mass and momentum. The fluid-dynamical field on either side of the front was assumed to be inviscid and incompressible. The structure of the flame, due to the complex interaction of chemical reaction(s), species diffusion and heat conduction, was completely ignored in this model. Instead, an expression for the flame velocity, defined as the normal component of the velocity of the unburned gas mixture relative to the moving surface, was prescribed.

Mathematically, the flame front is described by the function

$$F(\mathbf{X},t) = x - f(y,z,t) = 0. \tag{1.1}$$

On the surface

$$\frac{dF}{dt} = F_t + \text{grad}\, F \,.\, \frac{d\mathbf{X}}{dt} = 0, \tag{1.2}$$

where $\boldsymbol{v} = d\mathbf{X}/dt$ is the velocity of the surface. Then the normal velocity v_n of the surface element is given by

$$v_n \equiv \boldsymbol{v} \,.\, \mathbf{n} = \frac{-F_t}{|\text{grad}\, F|} = \frac{f_t}{(1+|\nabla f|^2)^{\frac{1}{2}}}, \tag{1.3}$$

where the unit normal $\mathbf{n}$ is given by $\operatorname{grad} F/|\operatorname{grad} F|$, and $\nabla = (\partial/\partial y, \partial/\partial z)$ is the two-dimensional gradient. Writing the velocity field $\mathbf{V}$ as $\mathbf{V} = u\mathbf{i}+\mathbf{v}$, where $\mathbf{v}$ is its two-dimensional transverse component, the flame speed S_f is thus given by

$$S_f \equiv \mathbf{V}.\mathbf{n}-\nu_n = \frac{u-\mathbf{v}.\nabla f-f_t}{(1+|\nabla f|^2)^{\frac{1}{2}}}, \tag{1.4}$$

where $\mathbf{V}$ in (1.4) is evaluated just ahead of the front, i.e. at $x = f^-$.

The jump conditions, corresponding to continuity of the tangential velocity components, as well as conservation of mass and momentum, are given respectively by

$$[\![\mathbf{V}\times\mathbf{n}]\!] = 0, \tag{1.5}$$

$$[\![\rho(\mathbf{V}.\mathbf{n}-\nu_n]\!] = 0, \tag{1.6}$$

$$[\![p+\rho\mathbf{V}.\mathbf{n}(\mathbf{V}.\mathbf{n}-\nu_n]\!] = 0, \tag{1.7}$$

where ρ and p denote respectively the density and pressure of the gas mixture, and $[\![\phi]\!] \equiv \phi(x=f^+)-\phi(x=f^-)$ denotes the jump in the quantity ϕ across the front. Finally one must specify the density jump $[\![\rho]\!]$ across the front. A complete description of the flow field on both sides of the flame can now be given in principle by solving Euler's equation subject to the conditions (1.5)–(1.7) provided that S_f is known.

The earliest treatments of flames according to the model just described are due to Landau (1944) and Darrieus (1945). In analysing the stability properties of plane flames, they made the simplest assumption by choosing S_f to be a prescribed constant. If velocities are normalized with respect to the propagation velocity of a plane adiabatic flame, the Landau–Darrieus model consists of (1.5)–(1.7) and

$$S_f = 1. \tag{1.8}$$

Owing to the heat released during combustion, $[\![\rho]\!] < 0$, and the Landau–Darrieus analysis predicts that plane flames are unconditionally unstable. Indeed they obtain that the growth rate of a disturbance of wavenumber k is given by

$$\omega = \omega_0 k, \tag{1.9}$$

with $\omega_0 > 0$ (see figure 2). However, this result is not in accord with observations, since plane flames are observed in the laboratory. Later works attempted to improve on the Landau–Darrieus model by assuming different expressions for S_f. Markstein (1951) suggested a dependence of S_f on flame-front curvature, by introducing a phenomenological parameter independent of the physicochemical parameters of the problem. This study, as well as later related work by Eckhaus (1961) and others, is described in Markstein (1964).

In this paper we consider the flame to consist of a thin boundary layer where transport processes dominate and in which there is another, much thinner reactive boundary layer (see figure 3). If l_D represents the characteristic length associated with diffusion and L a typical length of the problem, e.g. the scale of the outer fluid-dynamical field, then the flame thickness is $O(\delta)$, where $\delta = l_D/L$. The reactive boundary layer is $O(\epsilon\delta)$, where ϵ is inversely proportional to the activation energy E of the chemical reaction. Many of the reactions occurring in combustion in fact have large activation energies. Thus for $E \gg 1$ the reaction rate is strongly temperature-dependent, and the chemical reaction is confined to a thin reactive diffusive layer. Typical flames have reaction zones $\sim 10^{-3}$ mm and transport zones $\sim 10^{-1}$ mm. Thus on the fluid-dynamical scale the flame may be regarded as a moving front. In order to account for the interaction of chemical-reaction and transport

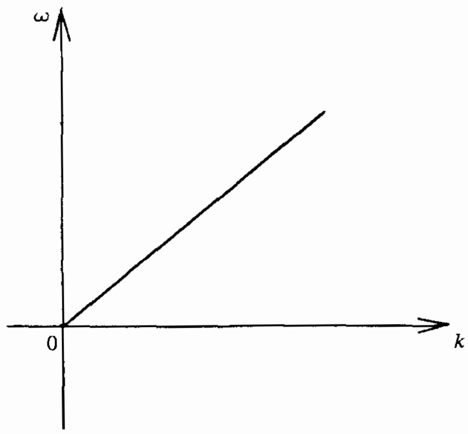

FIGURE 2. Growth rate of a disturbance versus its wavenumber according to results of Landau and Darrieus.

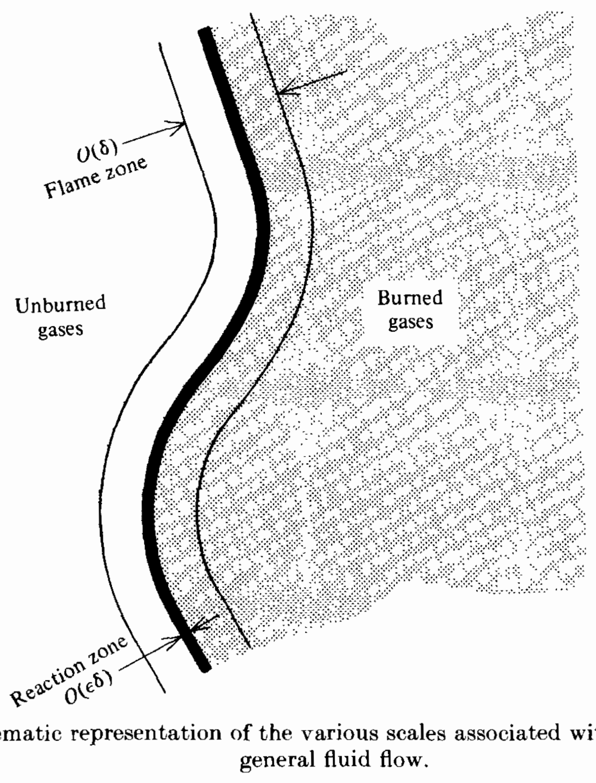

FIGURE 3. Schematic representation of the various scales associated with a curved flame in a general fluid flow.

processes with the fluid flow, we analyse the structure of the flame by determining the solution inside each boundary layer. We employ the method of matched asymptotic expansions to match the boundary layer solutions to the outer flows, and derive, rather than prescribe, the appropriate jump conditions across the front as well as an equation for the evolution of the shape and location of the front $f(y,z,t)$. The solution of the problem is given as an asymptotic expansion in powers of δ. The leading terms in the expansion correspond to the Landau–Darrieus model, while higher-order terms, which explicitly depend on the physico-chemical parameters account for the structure of the flame and provide corrections to their model. The corrected model has been employed to reconsider the stability problem for a plane flame. Finally we mention that there have been attempts to use the Landau–Darrieus model with the constant-flame-speed assumption in order to simulate flame propagation numerically in a variety of fluid flows. Thus our results also provide more accurate prescriptions for the velocity of, and the jump conditions across, the front for such simulations.

To be sure, in recent notable works Sivashinsky (1976), Clavin & Williams (1982) and Pelce & Clavin (1982) have also analysed the flame structure and its interaction with the fluid flow. Sivashinsky (1976) considered flames that were slowly varying in both space and time and assumed that f varies on the slower scale in order to permit $O(1)$ flame deformations. In contrast to our scaling by L, he scaled lengths by l_{D} and considered an $O(\epsilon)$ reaction zone, an $O(1)$ transport zone, and an $O(\epsilon^{-1})$ far-field fluid zone. Thus he effectively took $\delta = \epsilon$, and derived an expression for S_{f} and jump conditions identical with (1.5)–(1.7). With a similar non-dimensionalization to that in Sivashinsky, Clavin & Williams (1982) used a multiscale method in which they effectively employed the scales ϵ and δ. They further assumed both f and $\mathbf{v}$ to be $O(\delta)$ and derived an evolution equation for f. Under the same assumptions, Pelce & Clavin (1982) following Clavin & Williams derived jump conditions across the flame front. In contrast, in our analysis both f and $\mathbf{v}$ are taken to be $O(1)$, so that our results are valid for arbitrary flame shapes in general fluid flows. In this sense the analyses of Clavin & Williams and Pelce & Clavin may be considered a linearization of our results and can be recovered if f and $\mathbf{v}$ are specialized to be $O(\delta)$.

In §2 we non-dimensionalize and scale the governing equations. In §3 we analyse the reactive boundary layer, while in §4 we describe the outer fluid flow in which the flame is considered as a surface discontinuity. In §5 we analyse the diffusive transport boundary layer, and match to the outer fluid flow to obtain the jump conditions and the evolution equation. Finally, in §6 we summarize our results and draw conclusions. Specifically, we discuss the motion of the flame front (§6.1), the jump conditions across the flame (§6.2), the vorticity production in the flame (§6.3), the temperature of the burned gas (§6.4), and the stability of plane flames (§6.5).

2. Governing equations

We consider a homogeneous premixed combustible mixture of density $\rho_{-\infty}$ and temperature $T_{-\infty}$ that undergoes a one-step irreversible exothermic chemical reaction. In the reactive mixture, one of the components, say M_0, appears in sufficiently large quantity so that all physical properties of the mixture are determined essentially by that component. For convenience we further assume that the mixture contains a single deficient component M_1 and that the rate of progress of the chemical reaction depends on M_1 alone. Thus when M_1 is depleted the chemical reaction terminates. Further, we assume that M_1 appears in sufficiently small quantities so that it is only necessary to follow its evolution rather than the evolution of the other components whose concentration remains relatively unchanged. A mass balance for M_1 provides a single diffusion equation that accounts for the rate of its consumption. More generally, if we were to follow j reactants, then j diffusion equations result. In addition, mass, momentum and energy equations for the whole mixture must be considered simultaneously, the latter being modified to account for the heat released by the chemical reaction.

With the exception of velocities, we non-dimensionalize all variables with respect to their values in the fresh cold mixture. Thus $T_{-\infty}$, $\rho_{-\infty}$ and $p_{-\infty}$ are used as units of temperature, density and pressure respectively, and $Y_{-\infty}$ as a unit for the mass fraction of M_1. Velocities are non-dimensionalized by S_f^0, the propagation velocity of a plane adiabatic flame throughout the given mixture. The diffusion process introduces a lengthscale $l_D = \lambda/\rho_{-\infty} c_p S_f^0$, where λ is the thermal conductivity of the mixture of c_p its specific heat at constant pressure. However, rather than using l_D as the characteristic length, as did Sivashinsky (1976) and Clavin & Williams (1982), we non-dimensionalize distances with respect to a characteristic dimension L of the gasdynamic field, such as a typical radius of curvature of a curved flame. The ratio

$$\delta \equiv \frac{l_D}{L} \tag{2.1}$$

represents the relative thickness of the transport zone, or the flame. Finally the time variable is non-dimensionalized by L/S_f^0.

The non-dimensional governing equations are then given by

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0, \tag{2.2}$$

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\,.\,\mathrm{grad})\,\mathbf{V}\right) = -\mathrm{grad}\, p + \delta Pr\left\{\tilde{\Delta}\mathbf{V} + \frac{1}{3}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{V}\right\}, \tag{2.3}$$

$$\rho\left(\frac{\partial Y}{\partial t} + \mathbf{V}\,.\,\mathrm{grad}\;Y\right) - \delta Le^{-1}\tilde{\Delta} Y = -\delta\Omega, \tag{2.4}$$

$$\rho\left(\frac{\partial T}{\partial t} + \mathbf{V}\,.\,\mathrm{grad}\, T\right) - \delta\tilde{\Delta}_{\mathrm{T}} = \delta q \Omega, \tag{2.5}$$

where ρ is the density of the mixture and T its temperature, Y is the mass fraction of species M_1, and $\mathbf{V}$ the velocity field. Equations (2.2)–(2.5) are written under the assumption that flames propagate at speeds much smaller than the speed of sound. Thus, the representative Mach number $Ma = S_f^0(\rho_{-\infty}/p_{-\infty})^{\frac{1}{2}}$ is very small, implying that the process is nearly isobaric. Since only pressure gradients appear in the momentum equation, p in (2.3) represents only small deviations from its uniform ambient value. The total pressure is therefore given by $1 + (Ma)^2 p$. As a consequence, the equation of state simplifies to

$$\rho T = 1, \tag{2.6}$$

and the viscous-dissipation terms normally appearing in the energy equation are absent in (2.5). The system (2.2)–(2.6) thus corresponds to the leading term in an expansion with respect to Ma. Another assumption made is that the coefficient of viscosity μ, the thermal conductivity λ, the specific heat c_p of the mixture as well as $\mathscr{D} = \rho_{-\infty} D_1$ (where D_1 is the diffusion coefficient of M_1) are all constant. Therefore the parameters appearing in the equations are the Prandtl number $Pr = \mu c_p/\lambda$ representing the ratio of viscous to thermal diffusivities, the Lewis number $Le = \lambda/\mathscr{D} c_p$ representing the ratio of heat to mass diffusivities, the length ratio δ defined by (2.1), and the heat release q; the latter being the total heat of reaction per unit mass of reactant consumption made dimensionless by $c_p T_{-\infty}$. Finally, the reaction rate Ω, assumed to be of Arrhenius type takes the form

$$\Omega = \Lambda \delta^{-2} \rho Y \exp\left(-\frac{E}{T}\right), \tag{2.7}$$

where E is the activation energy of the reaction made dimensionless by $RT_{-\infty}$, R being the universal gas constant. The preexponential factor Λ in (2.7) is chosen so that the non-dimensional flame speed of a plane adiabatic flame is unity.

The governing equations (2.2)–(2.6) are to be solved subject to appropriate initial and boundary conditions. In the following we consider the case of an already established flame propagating into the fresh mixture. Thus for example we choose the initial conditions so that they are close to those that correspond to the established propagating flame rather than considering the evolution of arbitrary initial data to form the flame. In particular, we assume that the initial distribution of Y is zero behind the flame, corresponding to the fact that the reactant M_1 is totally depleted by the chemical reaction. Then this property will be preserved for all time. The velocity field and pressure distribution will be kept arbitrary in our formulation so that no boundary conditions will be imposed on $\mathbf{V}$ and p. However, since it has been assumed that the mixture is homogeneous and of constant composition, the mass fraction of M_1 and the temperature of the mixture are prescribed constants far upstream in the fresh mixture. In dimensionless form

$$Y = T = 1 \quad \text{as} \quad x \to -\infty \tag{2.8}$$

for all times $t \geqslant 0$.

The nonlinear flow field described by (2.2)–(2.3) is further complicated by the transport equations (2.4)–(2.5). A simplification often adopted is that the flow is not affected by the flame. However, this simplification can only be justified when the heat release q is small (Matkowsky & Sivashinsky 1979). Although such an approach has proven to be useful for qualitative understanding of flame propagation, it is not necessarily a common characteristic of actual flames. In the following, we consider q to be $O(1)$, thus taking into account the full interaction between the flame and the gas through which it propagates. As discussed in §1, an asymptotic analysis in which the activation energy E is large and the length ratio δ is small will be carried out. For $\delta \ll 1$, the reaction and transport processes are confined to a thin region that we call the flame. For $E \gg 1$, the reaction rate is appreciable only in a thin zone contained inside the flame and may be neglected elsewhere. In the limit $\delta \to 0$, the flame, together with the reaction zone embedded in it, shrinks to a moving surface – the flame front. The problem then reduces to the study of the fluid flow on both sides of the flame front. However, the analysis of the flame structure is essential in order to relate the properties of the fluid variables across the flame and to describe the instantaneous shape and motion of the flame front.

It is convenient to adopt a coordinate system attached to the flame front. Thus, if the moving front is described by (1.1), we introduce

$$\xi = x - f(y,z,t), \quad y = y, \quad z = z, \quad t = t. \tag{2.9}$$

Equations (2.2)–(2.5) now take the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \xi}(\rho s) + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.10}$$

$$\rho\frac{\partial u}{\partial t} + \rho s\frac{\partial u}{\partial \xi} + \rho \mathbf{v} \cdot \nabla u = -\frac{\partial p}{\partial \xi} + \delta Pr\left\{\Delta u + \frac{1}{3}\frac{\partial}{\partial \xi}\left(\frac{\partial s}{\partial \xi} + \nabla \cdot \mathbf{v}\right)\right\}, \tag{2.11}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho s\frac{\partial \mathbf{v}}{\partial \xi} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla f\frac{\partial p}{\partial \xi} + \delta Pr\left\{\Delta \mathbf{v} + \frac{1}{3}\left(\nabla - \nabla f\frac{\partial}{\partial \xi}\right)\left(\frac{\partial s}{\partial \xi} + \nabla \cdot \mathbf{v}\right)\right\}, \tag{2.12}$$

$$\rho\frac{\partial Y}{\partial t}+\rho s\frac{\partial Y}{\partial \xi}+\rho\mathbf{v}\,.\,\nabla Y-\delta Le^{-1}\Delta Y=-\delta\Omega, \tag{2.13}$$

$$\rho\frac{\partial T}{\partial t}+\rho s\frac{\partial T}{\partial \xi}+\rho\mathbf{v}\,.\,\nabla T-\delta\Delta T=q\delta\Omega, \tag{2.14}$$

where s is the longitudinal velocity in the moving frame defined by

$$s\equiv u-f_t-\mathbf{v}\,.\,\nabla f, \tag{2.15}$$

and the Laplacian Δ in this coordinate system is given by

$$\Delta=(1+|\nabla f|^2)\frac{\partial^2}{\partial \xi^2}+\nabla^2-\nabla^2 f\frac{\partial}{\partial \xi}-2\frac{\partial}{\partial \xi}(\nabla f\,.\,\nabla).$$

It should be noted that the velocity field is $\mathbf{V}=u\mathbf{i}+\mathbf{v}$ and that ∇ is the two-dimensional transverse gradient, as defined earlier. The flame front located at $\xi=0$ separates the fresh unburned gases in the region $\xi<0$, from the combustion products in the region $\xi>0$ where $Y\equiv 0$. Finally, for convenience we introduce the notation

$$m\equiv\rho s \tag{2.16}$$

for the longitudinal mass flux.

3. The reaction zone

We first consider the limit $E\to\infty$ without any assumption on the magnitude of δ. Thus $\xi=0$ represents the reaction front alone. It is convenient to rewrite the reaction term (2.7) in the form

$$\Omega=\Lambda'\rho Y\exp\left\{\frac{T_a^2}{\epsilon}\left(\frac{1}{T_a}-\frac{1}{T}\right)\right\}, \tag{3.1}$$

with $\epsilon=T_a^2/E\ll 1$ and $\Lambda'=\Lambda\delta^{-2}\exp(-T_a/\epsilon)$, where $T_a=1+q$ is the adiabatic flame temperature, i.e. the flame temperature of a plane adiabatic flame. It is clear that from (3.1) that for large activation energy, i.e. for $\epsilon\ll 1$, the reaction rate Ω is negligibly small except in a thin $O(\epsilon)$ region that shrinks to the surface $\xi=0$ in the limit $\epsilon\to 0$. The temperature along this surface to leading order is $T_a=1+q$.

In real gas mixtures, the Lewis number is often near unity. Thus it is reasonable to write

$$Le=1+\epsilon l, \tag{3.2}$$

where l is $O(1)$. Another assumption employed in our analysis is that temperature gradients in the region of the burned gases are relatively small, i.e.

$$\frac{\partial T}{\partial \xi}=o(1)\quad(\xi>0). \tag{3.3}$$

Then the temperature of the burned gases remains everywhere and at all times within $O(\epsilon)$ of the adiabatic flame temperature $1+q$. From (2.13), (2.14) and with the assumption (3.2) we find that the enthalpy

$$\tilde{H}\equiv T+qY \tag{3.4}$$

associated with the reactant M_1 must satisfy the equation

$$\rho\frac{\partial \tilde{H}}{\partial t}+\rho s\frac{\partial \tilde{H}}{\partial \xi}+\rho\mathbf{v}\,.\,\nabla\tilde{H}-\delta\Delta\tilde{H}=-\epsilon\delta lq\Delta Y. \tag{3.5}$$

Since $Y \equiv 0$ behind the flame, the constant $1+q$ is the leading term of the expansion of $\tilde{H}$ in powers of ϵ, for $\xi > 0$. In addition it is the leading term of the solution for $\xi < 0$ that satisfies the boundary conditions (2.8). Thus we expand $\tilde{H}$ and T as

$$\tilde{H} = 1+q+\epsilon h(\xi, y, z, t)+O(\epsilon^2), \tag{3.6}$$

$$T = \theta(\xi, y, z, t)+O(\epsilon), \tag{3.7}$$

and substituting (3.4) for Y on the right-hand side of (3.5), we obtain for $\xi \gtrless 0$

$$\rho\frac{\partial h}{\partial t}+\rho s\frac{\partial h}{\partial \xi}+\rho \mathbf{v} \,.\, \nabla h-\delta\Delta h = \delta l\Delta\theta, \tag{3.8}$$

$$\rho\frac{\partial \theta}{\partial t}+\rho s\frac{\partial \theta}{\partial \xi}+\rho \mathbf{v} \,.\, \nabla \theta-\delta\Delta \theta = 0, \tag{3.9}$$

which replace (2.13) and (2.14) respectively. In particular we note that

$$\theta \equiv 1+q \quad (\xi \geqslant 0), \tag{3.10}$$

and that for the burned gases, the temperature deviations from the value (3.10) are expressed by h, since $Y \equiv 0$ there.

The analyses of the reaction-zone structure will not be repeated here since it follows closely that described by Matkowsky & Sivashinsky (1979), for the transport equations (3.8), (3.9). The results consist of the following jump conditions:

$$[\theta] = [h] = 0, \tag{3.11}$$

$$\delta(1+|\nabla f|^2)^{\frac{1}{2}}\left[\frac{\partial \theta}{\partial \xi}\right] = -q \exp\{\tfrac{1}{2}h(0, y, z, t)\}, \tag{3.12}$$

$$\left[\frac{\partial h}{\partial \xi}\right] = -l\left[\frac{\partial \theta}{\partial \xi}\right], \tag{3.13}$$

to be satisfied across the reaction front, at $\xi = 0$. Here $[\phi] = \phi(\xi = 0^+)-\phi(\xi = 0^-)$ denotes the jump in the variable ϕ across the reaction zone. As for the fluid variables, since (2.10)–(2.12) do not contain the reaction-rate term $\mathbf{\Omega}$, a direct integration across $\xi = 0$ yields

$$[u] = [\mathbf{v}] = [s] = 0, \tag{3.14}$$

$$[p] = \tfrac{4}{3}Pr\,\delta(1+|\nabla f|^2)\left[\frac{\partial u}{\partial \xi}\right]. \tag{3.15}$$

The continuity equation (2.10) also gives

$$\left[\frac{\partial s}{\partial \xi}\right] = m(O, y, z, t)\left[\frac{\partial \theta}{\partial \xi}\right], \tag{3.16}$$

where use has been made of (2.16). Summing (2.11) and (2.12) after multiplying the former by ∇f, we obtain

$$\left[\frac{\partial \mathbf{v}}{\partial \xi}+\nabla f\frac{\partial u}{\partial \xi}\right] = 0. \tag{3.17}$$

Finally, using (2.15) and (3.17) we write

$$\left[\frac{\partial s}{\partial \xi}\right] = (1+|\nabla f|^2)\left[\frac{\partial u}{\partial \xi}\right]. \tag{3.18}$$

We conclude that on any scale larger than $O(\epsilon)$, the reaction zone may be regarded as a surface of discontinuity ($\xi = 0$), thus effectively replacing the nonlinear reaction rate Ω by the jump conditions (3.11)–(3.18). The resulting problem consists of (2.10)–(2.12) for the fluid variables u, $\mathbf{v}$, p, and (3.8), (3.9) for the transport variables h, θ. Although these equations do not contain ϵ explicitly, the dependence of the physical variables $\tilde{H}$ and T on ϵ is manifested in (3.6), (3.7).

4. The flame as a surface of discontinuity

We now exploit the smallness of the parameter δ. Clearly, the limit $\delta \to 0$ is singular, since δ multiplies the highest derivatives in the governing equations, representing viscous dissipation, heat conduction and species diffusion. It indicates the existence of a thin transport zone that we have called the flame, where large gradients are confined. In the limit $\delta \to 0$, the flame and the reaction zone embedded in it shrink to the surface $\xi = 0$. Then, on both sides of the flame, i.e. for $\xi \lessgtr 0$, we seek outer expansions of the form

$$\left.\begin{aligned} \theta &= \Theta_0+\delta\Theta_1+\dots, & h &= H_0+\delta H_1+\dots,\\ \rho &= R_0+\delta R_1+\dots, & s &= S_0+\delta S_1+\dots,\\ u &= U_0+\delta U_1+\dots, & \mathbf{v} &= \mathbf{V}_0+\delta \mathbf{V}_1+\dots,\\ p &= P_0+\delta P_1+\dots, & f &= f^0+\delta f^1+\dots. \end{aligned}\right\} \tag{4.1}$$

We first consider the transport equations (3.8), (3.9). The leading terms of the expansions (4.1) satisfy

$$\frac{\partial \Theta_0}{\partial t}+S_0\frac{\partial \Theta_0}{\partial \xi}+\mathbf{V}_0\,.\,\nabla\Theta_0 = 0, \tag{4.2}$$

$$\frac{\partial H_0}{\partial t}+S_0\frac{\partial H_0}{\partial \xi}+\mathbf{V}_0\,.\,\nabla H_0 = 0. \tag{4.3}$$

That is, the convective derivative (in the moving coordinates) of Θ_0 and H_0 vanishes, implying that both do not vary along particle paths in each region. Since according to (2.8) $\Theta_0 = R_0 = 1$, $H_0 = 0$ are the given values far upstream, then

$$\Theta_0 = R_0 \equiv 1, \quad H_0 \equiv 0 \quad (\xi < 0). \tag{4.4}$$

Furthermore, (4.4) is the solution valid to all orders in δ, since all perturbations of Θ, R and h must vanish as $\xi \to -\infty$. According to (3.10) we conclude that

$$\Theta_0 \equiv R_0^{-1} \equiv 1+q \quad (\xi > 0), \tag{4.5}$$

which is also the solution to all orders in δ. However, nothing can be said yet about the quantities H_i for $\xi > 0$, since the flame is expected to produce non-uniform temperature perturbations that are expressed by h. This information, contained in the values $h(0, y, z, t)$ will be extracted from the flame structure analysis.

We now consider the flow-field equations (2.10)–(2.12). Since on either side of the flame the density is a constant given by

$$R = \begin{cases} 1 & (\xi < 0) \\ (1+q)^{-1} & (\xi > 0), \end{cases}$$

to all orders in δ, the continuity and momentum equations simplify to

$$\frac{\partial s}{\partial \xi}+\nabla\,.\,\mathbf{v} = 0, \tag{4.6}$$

$$R\left(\frac{\partial u}{\partial t}+s\frac{\partial u}{\partial \xi}+\mathbf{v}\,.\,\nabla u\right) = -\frac{\partial p}{\partial \xi}+\delta Pr\Delta u, \tag{4.7}$$

$$R\left(\frac{\partial \mathbf{v}}{\partial t}+s\frac{\partial \mathbf{v}}{\partial \xi}+(\mathbf{v}\,.\,\nabla)\,\mathbf{v}\right) = -\nabla p+\nabla f\frac{\partial p}{\partial \xi}+\delta Pr\Delta \mathbf{v}. \tag{4.8}$$

Substituting the expansions (4.1) into (4.6)–(4.8) results in a series of first-order equations valid for $\xi \lessgtr 0$ to each order in δ. To leading order the flow field satisfies Euler's equations

$$\frac{\partial S_0}{\partial \xi}+\nabla\,.\,\mathbf{V}_0 = 0, \tag{4.9a}$$

$$R\left(\frac{\partial U_0}{\partial t}+S_0\frac{\partial U_0}{\partial \xi}+\mathbf{V}_0\,.\,\nabla U_0\right) = -\frac{\partial P_0}{\partial \xi}, \tag{4.9b}$$

$$R\left(\frac{\partial \mathbf{V}_0}{\partial t}+S_0\frac{\partial \mathbf{V}_0}{\partial \xi}+(\mathbf{V}_0\,.\,\nabla)\,\mathbf{V}_0)\right) = -\nabla P_0+\nabla f^0\frac{\partial P_0}{\partial \xi}, \tag{4.9c}$$

$$S_0 = U_0-f^0_t-\mathbf{V}_0\,.\,\nabla f^0; \tag{4.9d}$$

namely the incompressible inviscid equations with different density values on either side of the flame. Higher terms yield corrections, including those attributed to the small but non-negligible viscous forces. In particular

$$\frac{\partial S_1}{\partial \xi}+\nabla\,.\,\mathbf{V}_1 = 0, \tag{4.10a}$$

$$R\left(\frac{\partial U_1}{\partial t}+S_0\frac{\partial U_1}{\partial \xi}+S_1\frac{\partial U_0}{\partial \xi}+\mathbf{V}_0\,.\,\nabla U_1+\mathbf{V}_1\,.\,\nabla U_0\right) = -\frac{\partial P_1}{\partial \xi}+Pr\Delta^0 U_0, \tag{4.10b}$$

$$R\left(\frac{\partial \mathbf{V}_1}{\partial t}+S_0\frac{\partial \mathbf{V}_1}{\partial \xi}+S_1\frac{\partial \mathbf{V}_0}{\partial \xi}+(\mathbf{V}_0\,.\,\nabla)\,\mathbf{V}_1+(\mathbf{V}_1\,.\,\nabla)\,\mathbf{V}_0)\right)$$
$$= -\nabla P_1+\nabla f^0\frac{\partial P_1}{\partial \xi}+\nabla f^1\frac{\partial P_0}{\partial \xi}+Pr\Delta^0\mathbf{V}_0 \tag{4.10c}$$

$$S_1 = U_1-f^1_t-\mathbf{V}_0\,.\,\nabla f^1-\mathbf{V}_1\,.\,\nabla f^0, \tag{4.10d}$$

where the superscript in Δ^0 indicates that f in Δ must be replaced by f^0.

5. The flame structure

To study the flame structure, we introduce the stretching transformation

$$\xi = \delta\zeta \tag{5.1}$$

and seek inner expansions of the form

$$\left.\begin{array}{ll} \theta = \theta_0+\delta\theta_1+\ldots, & h = h_0+\delta h_1+\ldots, \\ \rho = \rho_0+\delta\rho_1+\ldots, & s = s_0+\delta s_1+\ldots, \\ u = u_0+\delta u_1+\ldots, & \mathbf{v} = \mathbf{v}_0+\delta\mathbf{v}_1+\ldots, \\ p = p_0+\delta p_1+\ldots. & m = m_0+\delta m_1+\ldots. \end{array}\right\} \tag{5.2}$$

Then the fluid equations (2.10)–(2.12), the transport equations (3.8), (3.9) and the jump conditions (3.11)–(3.18), which we rewrite in terms of ζ, yield a system of equations to be solved recursively for the coefficients of the expansions (5.2). This system describes the transport region, characterized as a convective diffusive zone, containing a thin reactive zone.

To leading order we have, for $-\infty < \zeta < \infty$,

$$\frac{\partial m_0}{\partial \zeta} = 0, \tag{5.3}$$

$$m_0 \frac{\partial \theta_0}{\partial \zeta} - (1+|\nabla f^0|^2)\frac{\partial^2 \theta_0}{\partial \zeta^2} = 0, \tag{5.4}$$

$$m_0 \frac{\partial h_0}{\partial \zeta} - (1+|\nabla f^0|^2)\frac{\partial^2 h_0}{\partial \zeta^2} = l(1+|\nabla f^0|^2)\frac{\partial^2 \theta_0}{\partial \zeta^2}, \tag{5.5}$$

$$m_0 \frac{\partial u_0}{\partial \xi} - Pr\left\{(1+|\nabla f^0|^2)\frac{\partial^2 u_0}{\partial \zeta^2} + \frac{1}{3}\frac{\partial^2 s_0}{\partial \zeta^2}\right\} = -\frac{\partial p_0}{\partial \zeta}, \tag{5.6}$$

$$m_0 \frac{\partial \mathbf{v}_0}{\partial \zeta} - Pr\left\{(1+|\nabla f^0|^2)\frac{\partial^2 \mathbf{v}_0}{\partial \zeta^2} - \frac{1}{3}\frac{\partial^2 s_0}{\partial \zeta^2}\nabla f^0\right\} = \nabla f^0 \frac{\partial p_0}{\partial \zeta}, \tag{5.7}$$

$$s_0 = u_0 - f_t^0 - \mathbf{v}_0 \cdot \nabla f^0, \tag{5.8}$$

$$m_0 = \rho_0 s_0, \quad \rho_0 \theta_0 = 1. \tag{5.9}$$

Equation (5.3) implies that m_0 is independent of ζ, i.e. $m_0 = m_0(y,z,t)$, and therefore solutions for θ_0 and h_0 are readily available. In particular they must satisfy the matching conditions $\theta_0 = 1$ and $h_0 = 0$ as $\zeta \to -\infty$ implied by (4.4), and in addition $\theta_0 \equiv 1+q$ for $\zeta \geqslant 0$ according to (3.10). Following (3.11), both θ_0 and h_0 are continuous at $\zeta = 0$, and the jumps in their derivatives, accounting for the effects of the reactive zone, satisfy

$$\left.\begin{aligned} (1+|\nabla f^0|^2)^{\frac{1}{2}}\left[\frac{\partial \theta_0}{\partial \zeta}\right] &= -q\exp\{\tfrac{1}{2}h_0(0,y,z,t)\}, \\ \left[\frac{\partial h_0}{\partial \zeta}\right] &= -l\left[\frac{\partial \theta_0}{\partial \zeta}\right]. \end{aligned}\right\} \tag{5.10}$$

Therefore we obtain

$$m_0 = (1+|\nabla f^0|^2)^{\frac{1}{2}} \tag{5.11}$$

$$\theta_0 = \begin{cases} 1+q\exp(\zeta/m_0) & (\zeta<0), & (5.12a) \\ 1+q & (\zeta>0), & (5.12b) \end{cases}$$

$$h_0 = \begin{cases} -lqm_0^{-1}\zeta\exp(\zeta/m_0) & (\zeta<0), & (5.13a) \\ 0 & (\zeta>0). & (5.13b) \end{cases}$$

In order to determine the velocity field we add (5.6) to (5.7) after multiplying the former by ∇f^0. According to (3.17) the combination $\mathbf{v}_0 + \nabla f^0 u_0$ is continuous and has continuous derivatives at $\zeta = 0$. Therefore, after discarding exponentially growing terms as unmatchable, we conclude that $\mathbf{v}_0 + \nabla f^0 u_0$ is independent of ζ. Using (5.8) and (5.9), and matching the results with the outer expansions (4.1), we obtain

$$u_0 = \begin{cases} U_0(0^-,y,z,t) + (q/m_0)\exp(\zeta/m_0) & (\zeta<0), & (5.14a) \\ U_0(0^-,y,z,t) + q/m_0 & (\zeta>0), & (5.14b) \end{cases}$$

$$\mathbf{v}_0 = \begin{cases} \mathbf{V}_0(0^-,y,z,t) - \nabla f^0(q/m_0)\exp(\zeta/m_0) & (\zeta<0), & (5.15a) \\ \mathbf{V}_0(0^-,y,z,t) - \nabla f^0(q/m_0) & (\zeta>0). & (5.15b) \end{cases}$$

Equation (5.8) then implies that

$$f_t^0 + \mathbf{V}_0(0^-, y, z, t) \,.\, \nabla f^0 - U_0(0^-, y, z, t) + (1 + |\nabla f^0|^2)^{\frac{1}{2}} = 0, \tag{5.16}$$

which indicates that to *leading order* the flame speed S_f defined by (1.4) is indeed unity, as assumed by Landau and Darrieus. Finally, the pressure distribution is obtained after integrating (5.6) once and using the jump condition

$$[p_0] = \tfrac{4}{3}Pr(1 + |\nabla f^0|^2)\left[\frac{\partial u_0}{\partial \zeta}\right]$$

at $\zeta = 0$. Matching with the outer pressure field then implies that

$$p_0 = \begin{cases} P_0(0^-, y, z, t) + (\tfrac{4}{3}Pr - 1)\, q \exp(\zeta/m_0) \quad (\zeta < 0), & (5.17a) \\ P_0(0^-, y, z, t) - q \quad (\zeta > 0). & (5.17b) \end{cases}$$

The jump conditions to *leading order* across the flame front can now be summarized as

$$\left.\begin{aligned} [\![U_0]\!] &= q(1 + |\nabla f^0|^2)^{-\frac{1}{2}}, \\ [\![\mathbf{V}_0 + U_0 \nabla f^0]\!] &= 0, \\ [\![P_0]\!] &= -q, \end{aligned}\right\} \tag{5.18}$$

where $[\![\phi]\!] = \phi(\xi = 0^+) - \phi(\xi = 0^-)$ denotes the jump in the variable ϕ across the flame. The quantity $[\![\phi]\!]$ represents the jump in ϕ across the transport zone, i.e. the flame, in contrast to the quantity $[\phi]$, which represents the jump in ϕ across the reaction zone. It can easily be seen that the conditions (5.18) are indeed identical with (1.5)–(1.7). Thus, in order to obtain corrections to the Landau–Darrieus model, we shall include the $O(\delta)$ terms in the analysis.

We first consider the continuity and transport equations

$$\frac{\partial m_1}{\partial \zeta} = -\frac{\partial \rho_0}{\partial t} - \nabla \,.\, (\rho_0 \mathbf{v}_0), \tag{5.19}$$

$$m_0 \frac{\partial \theta_1}{\partial \zeta} - m_0^2 \frac{\partial^2 \theta_1}{\partial \zeta^2} = -m_1 \frac{\partial \theta_0}{\partial \zeta} + 2\nabla f^0 \,.\, \nabla f^1 \frac{\partial^2 \theta_0}{\partial \zeta^2} - \rho_0\left(\frac{\partial \theta_0}{\partial t} + \mathbf{v}_0 \,.\, \nabla \theta_0\right) - \nabla^2 f^0 \frac{\partial \theta_0}{\partial \zeta} - 2\frac{\partial}{\partial \zeta}(\nabla f^0 \,.\, \nabla \theta_0), \tag{5.20}$$

$$m_0 \frac{\partial h_1}{\partial \zeta} - m_0^2 \frac{\partial^2 h_1}{\partial \zeta^2} = l m_0^2 \frac{\partial^2 \theta_1}{\partial \zeta^2} - m_1 \frac{\partial h_0}{\partial \zeta} - \rho_0\left(\frac{\partial h_0}{\partial t} + \mathbf{v}_0 \,.\, \nabla h_0\right) + 2\nabla f^0 \,.\, \nabla f^1\left(\frac{\partial^2 h_0}{\partial \zeta^2} + l\frac{\partial^2 \theta_0}{\partial \zeta^2}\right) - \nabla^2 f^0\left(\frac{\partial h_0}{\partial \zeta} + l\frac{\partial \theta_0}{\partial \zeta}\right) - 2\frac{\partial}{\partial \zeta}(\nabla f^0 \,.\, \nabla h_0 + l\nabla f^0 \,.\, \nabla \theta_0), \tag{5.21}$$

to be solved for $-\infty < \zeta < \infty$. At $\zeta = 0$, the quantities m_1, θ_1 and h_1 are all continuous, and according to (3.12), (3.13),

$$(1 + |\nabla f^0|^2)^{\frac{1}{2}}\left[\frac{\partial \theta_1}{\partial \zeta}\right] = -\tfrac{1}{2} q h_1(0, y, z, t) + q\frac{\nabla f^0 \,.\, \nabla f^1}{1 + |\nabla f^0|^2}, \tag{5.22a}$$

$$\left[\frac{\partial h_1}{\partial \zeta}\right] = -l\left[\frac{\partial \theta_1}{\partial \zeta}\right], \tag{5.22b}$$

which accounts for the effects of the reaction zone. The resulting solution for m_1 is

$$m_1 = G(y,z,t) - \zeta\nabla\cdot\mathbf{V}_0(0^-,y,z,t) - m_0^{-1}\frac{\tilde{D}m_0}{Dt}\frac{q\zeta e^{\zeta/m_0}}{1+qe^{\zeta/m_0}}$$
$$+\left(\frac{Dm_0}{Dt} + m_0\nabla\cdot\mathbf{V}_0(0^-,y,z,t) + \nabla^2 f^0\right)\ln\left(1+qe^{\zeta/m_0}\right) \quad (\zeta<0), \qquad (5.23a)$$

$$m_1 = G(y,z,t) - \frac{\zeta}{1+q}\left(\nabla\cdot\mathbf{V}_0(0^-,y,z,t) - q\nabla^2 f^0/m_0 + q\frac{\nabla f^0\cdot\nabla m_0}{m_0^2}\right)$$
$$+\left(\frac{Dm_0}{Dt} + m_0\nabla\cdot\mathbf{V}_0(0^-,y,z,t) + \nabla^2 f^0\right)\ln(1+q) \quad (\zeta>0), \qquad (5.23b)$$

where G, as yet an undetermined function, is independent of ζ, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}_0(0^-,y,z,t)\cdot\nabla, \quad \frac{\tilde{D}}{Dt} \equiv \frac{D}{Dt} + \frac{\nabla f^0\cdot\nabla}{m_0} \qquad (5.24)$$

are the convective derivatives associated with the transverse velocity vector just ahead of the flame. This notation will also be used in the subsequent analysis. For $\zeta > 0$, the right-hand sides of (5.20) and (5.21) vanish, implying that θ_1 and h_1 are independent of ζ since exponentially growing terms must be discarded as unmatchable. In particular, since the outer solution (4.5) for θ is valid to all others in δ, matching implies that $\theta_1 \equiv 0$ for $\zeta > 0$. Therefore

$$\theta_1 \equiv 0, \quad h_1 \equiv h_1(0,y,z,t) \quad (\zeta > 0). \qquad (5.25)$$

Although the solutions θ_1 and h_1 for $\zeta < 0$ can be obtained, all that is needed at the present is the integration of (5.20), (5.21) with respect to ζ from $-\infty$ to 0 in order to determine $\partial\theta_1/\partial\zeta$ and $\partial h_1/\partial\zeta$ at $\zeta = 0^-$. Using (5.25) and the jump conditions (5.22) we obtain

$$m_0 h_1(0,y,z,t) = -lI\left(\nabla^2 f^0 + m_0\nabla\cdot\mathbf{V}_0(0^-,y,z,t) + \frac{Dm_0}{Dt}\right), \qquad (5.26)$$

$$G(y,z,t) = -\alpha\left(\nabla^2 f^0 + m_0\nabla\cdot\mathbf{V}_0(0^-,y,z,t) + \frac{Dm_0}{Dt}\right) + \frac{\nabla f^0\cdot\nabla f^1}{m_0}, \qquad (5.27)$$

where I, which depends only on q, is given by

$$I(q) = \int_{-\infty}^{0}\ln(1+qe^x)\,dx, \qquad (5.28)$$

and
$$\alpha \equiv \frac{1+q}{q}\ln(1+q) + \tfrac{1}{2}lI. \qquad (5.29)$$

We now consider the momentum equations for the velocities u_1 and $\mathbf{v}_1$ and the pressure p_1. Rather than writing the lengthy individual equations, we note that the combination $\mathbf{v}_1 + u_1\nabla f^0$ satisfies

$$m_0\frac{\partial}{\partial\zeta}(\mathbf{v}_1 + u_1\nabla f^0) - Pr\,m_0^2\frac{\partial}{\partial\zeta^2}(\mathbf{v}_1 + u_1\nabla f^0) = -\rho_0\left(\frac{\partial}{\partial t} + \mathbf{v}_0\cdot\nabla\right)(\mathbf{v}_0 + u_0\nabla f^0)$$
$$+\rho_0 u_0\left(\frac{\partial}{\partial t} + \mathbf{v}_0\cdot\nabla\right)\nabla f^0 + Pr\nabla(m_0^2)\frac{\partial u_0}{\partial\zeta} + \left(\nabla f^1\frac{\partial}{\partial\zeta} - \nabla\right)\left(p_0 - \frac{Pr}{3}\frac{\partial s_0}{\partial\zeta}\right) \qquad (5.30)$$

for $-\infty < \zeta < \infty$. According to (3.14) and (3.17), at $\zeta = 0$

$$[\mathbf{v}_1 + u_1 \nabla f^0] = 0, \tag{5.31a}$$

$$\left[\frac{\partial \mathbf{v}_1}{\partial \zeta} + \frac{\partial u_1}{\partial \zeta} \nabla f^0\right] = \frac{q \nabla f^1}{m_0^2}. \tag{5.31b}$$

We will avoid writing the complete solution, since our interest is in the determination of its behaviour as $\zeta \to \pm\infty$. In particular, after matching with two terms of the outer expansions (4.1), we obtain

$$\left[\!\!\left[\frac{\partial \mathbf{V}_0}{\partial \xi} + \frac{\partial U_0}{\partial \xi} \nabla f^0\right]\!\!\right] = \frac{q}{1+q}\left(\frac{1}{m_0} \frac{\tilde{D}}{Dt} \mathbf{V}_0(0^-, y, z, t) + \frac{\nabla f^0}{m_0} \frac{\tilde{D}}{Dt} U_0(0^-, y, z, t) + \frac{1}{m_0^2} \frac{\tilde{D}}{Dt} \nabla f^0\right) - \frac{q \nabla m_0}{m_0^2}, \tag{5.32}$$

$$[\![\mathbf{V}_1 + U_1 \nabla f^0]\!] = m_0 \sigma \left[\!\!\left[\frac{\partial \mathbf{V}_0}{\partial \xi} + \frac{\partial U_0}{\partial \xi} \nabla f^0\right]\!\!\right] - q\left(\frac{\nabla f^1}{m_0} - \sigma \frac{\nabla m_0}{m_0}\right), \tag{5.33}$$

where
$$\sigma \equiv Pr + \frac{1+q}{q} \ln(1+q). \tag{5.34}$$

It should be noted that, unlike (5.33), (5.32) does not yield any new information as it can be also derived directly from the outer equations (4.9). Employing (2.15) and (2.16) we obtain

$$m_0^2 u_1 = s_1 + f_t^1 + \mathbf{v}_0 . \nabla f^1 + \nabla f^0 (\mathbf{v}_1 + u_1 \nabla f^0), \tag{5.35}$$

$$s_1 = m_0 \theta_1 + m_1 \theta_0, \tag{5.36}$$

where use has been made of the equation of state $\rho_0 \theta_1 + \rho_1 \theta_0 = 0$. These equations provide the solution for u_1 in terms of already determined quantities. In particular the behaviour of u_1 as $\zeta \to \pm\infty$ can be determined. After matching with two terms of the outer expansions (4.1) we obtain

$$\left[\!\!\left[\frac{\partial U_0}{\partial \xi}\right]\!\!\right] = \nabla . \left(\frac{q \nabla f^0}{m_0}\right) + \left[\!\!\left[\nabla f^0 . \frac{\partial \mathbf{V}_0}{\partial \xi}\right]\!\!\right], \tag{5.37}$$

$$[\![U_1]\!] = -\frac{\gamma}{m_0^2}\left(\nabla^2 f^0 + m_0 \nabla . \mathbf{V}_0(0^-, y, z, t) + \frac{Dm_0}{Dt}\right) - \frac{q \nabla f^0 . \nabla f^1}{m_0^3} + \sigma\left(m_0 \left[\!\!\left[\frac{\partial U_0}{\partial \xi}\right]\!\!\right] - \frac{q \nabla^2 f^0}{m_0^2} + \frac{2q(\nabla m_0 . \nabla f^0)}{m_0^3}\right), \tag{5.38}$$

where
$$\gamma \equiv \tfrac{1}{2} l q I. \tag{5.39}$$

Again, (5.37) can be derived directly from the outer equations (4.9) and contains no new information. Matching also implies

$$f_t^1 + \mathbf{V}_0(0^-, y, z, t) . \nabla f^1 + \mathbf{V}_1(0^-, y, z, t) . \nabla f^0 - U_1(0^-, y, z, t) + \frac{\nabla f^0 . \nabla f^1}{m_0} = \alpha\left(\nabla^2 f^0 + m_0 \nabla . \mathbf{V}_0(0^-, y, z, t) + \frac{Dm_0}{Dt}\right). \tag{5.40}$$

This equation provides the $O(\delta)$ corrections to the flame-front evolution expressed in terms of f^1. Upon dividing both sides by m_0, and employing (1.4), it is easily seen that the left-hand side represents the $O(\delta)$ correction to the flame speed S_f. Thus (5.40) expresses the correction to the flame speed which is no longer constant, but varies along the front and with time. The physical significance of the various terms responsible for those variations in the flame speed will be discussed below.

Finally we consider the longitudinal component of the momentum equations

$$\frac{\partial p_1}{\partial \zeta} = -m_0\frac{\partial u_1}{\partial \zeta} - m_1\frac{\partial u_0}{\partial \zeta} - \rho_0\left(\frac{\partial u_0}{\partial t} + \mathbf{v}_0 . \nabla u_0\right) + Pr\left\{m_0^2\frac{\partial^2 u_1}{\partial \zeta^2}\right.$$
$$\left. + 2\nabla f^0 . \nabla f^1\frac{\partial^2 u_0}{\partial \zeta^2} - \nabla^2 f^0\frac{\partial u_0}{\partial \zeta} - 2\frac{\partial}{\partial \zeta}(\nabla f^0 . \nabla u_0) + \frac{1}{3}\frac{\partial}{\partial \zeta}\left(\frac{\partial s_1}{\partial \zeta} + \nabla . \mathbf{v}_0\right)\right\}, \quad (5.41)$$

to be solved for p_1. Equations (3.15) and (3.18) imply that

$$[p_1] = \tfrac{4}{3}Pr\left[\frac{\partial s_1}{\partial \zeta}\right]. \quad (5.42)$$

Integrating (5.41) with respect to ζ from $-\infty$ to $+\infty$, employing (5.42), and matching to the outer expansion (4.1), yields the jump in P_1 across the flame as

$$[\![P_1]\!] = -2m_0[\![U_1]\!] + (Pr+\sigma)\left\{m_0^2\left[\!\!\left[\frac{\partial U_0}{\partial \xi}\right]\!\!\right] - \frac{q\nabla^2 f^0}{m_0} + \frac{2q\nabla m_0 . \nabla f^0}{m_0^2}\right\}$$
$$+ q\frac{\nabla^2 f^0}{m_0} - 2q\frac{\nabla f^0 . \nabla f^1}{m_0^2} - q\frac{\nabla f^0 . \nabla m_0}{m_0^2} + \ln(1+q)$$
$$\times\left\{m_0\frac{DU_0}{Dt}(0^-, y, z, t) + \nabla f^0 . \nabla U_0(0^-, y, z, t) + \frac{1}{m_0}\frac{Dm_0}{Dt} + \frac{\nabla m_0 . \nabla f^0}{m_0^2}\right\}. \quad (5.43)$$

6. Summary of results

In this section we summarize the results we have derived from a study of the flame structure. In particular we discuss in turn results for the motion of the flame front, the jump conditions across the flame front, the vorticity production in the flame, the temperature of the burned gases, and the stability of plane flames. To present results valid to $O(\delta)$, we add together the results for the first two terms of the expansions (5.2).

6.1. *The motion of the flame front*

From (5.16) and (5.40) we obtain the equation governing the motion of the flame front as

$$\{u(0^-, y, z, t) - \mathbf{v}(0^-, y, z, t) . \nabla f - f_t\}N^{-1} = 1 - \delta\alpha\left\{\frac{\nabla^2 f}{N} + \nabla . \mathbf{v}(0^-, y, z, t) + \frac{1}{N}\frac{DN}{Dt}\right\} + o(\delta), \quad (6.1)$$

where $N = (1+|\nabla f|^2)^{\frac{1}{2}}$. Using (1.4), we can rewrite (6.1) as an equation for the flame speed, given by

$$S_f(y, z, t) = 1 - \delta\alpha\left\{\frac{\nabla^2 f}{N} + \nabla . \mathbf{v}(0^-, y, z, t) + \frac{1}{N}\frac{DN}{Dt}\right\} + o(\delta). \quad (6.2)$$

We observe that the correction terms to the Landau–Darrieus model are proportional to the flame thickness δ and to the parameter α defined by (5.29), which accounts for the heat release q and for differential diffusion l. The interpretation of the bracketed terms is related to the concept of flame stretch, first introduced by Karlovitz *et al.* (1953) as follows.

Let Δ represent the surface area of an element on the flame front. As the flame front moves through the fluid, the element deforms so that Δ varies with time. A measure of this deformation, or stretch, may be expressed by

$$\kappa \equiv \frac{1}{\Delta}\frac{d\Delta}{dt}. \quad (6.3)$$

Buckmaster (1979) derived an expression for the fractional variations of the area Δ, based on kinematic arguments. We can use his result to write an expression for κ in our notation, as

$$\kappa = \frac{1}{N}\left(\frac{\partial N}{\partial t} + \boldsymbol{\nu} . \nabla N\right) + \nabla . \boldsymbol{\nu} \tag{6.4}$$

where $\boldsymbol{\nu}$ is the velocity of the surface $\xi = 0$ with respect to a fixed frame of reference, as defined in §1. To relate κ to the flame speed S_f, we note that the surface element moves on the flame surface with velocity

$$\boldsymbol{\nu} - (\boldsymbol{\nu} . \mathbf{n})\,\mathbf{n} = \mathbf{V} - (\mathbf{V} . \mathbf{n})\,\mathbf{n}|_{\xi=0^-}, \tag{6.5}$$

that is, at a rate equal to the tangential component of the unburned gas velocity $\mathbf{V}$. Writing, as before, $\mathbf{V} = u\mathbf{i} + \mathbf{v}$ and using (1.3) and (1.4), as well as the expression $\mathbf{n} = N^{-1}(\mathbf{i} - \nabla f)$, we obtain

$$\boldsymbol{\nu} = \left(u(0^-, y, z, t) - \frac{1}{N} S_f\right)\mathbf{i} + \mathbf{v}(0^-, y, z, t) + \frac{S_f}{N}\nabla f. \tag{6.6}$$

Then (6.4) and (6.6) yield an implicit relation between flame stretch κ and flame speed S_f, given by

$$\kappa = \frac{1}{N}\nabla . (S_f \nabla f) + \nabla . \mathbf{v}(0^-, y, z, t) + \frac{1}{N}\frac{DN}{Dt}. \tag{6.7}$$

Then, since by (6.2) $S_f = 1 + O(\delta)$, we can rewrite (6.7) as

$$\kappa = \frac{1}{N}\nabla^2 f + \nabla . \mathbf{v}(0^-, y, z, t) + \frac{1}{N}\frac{DN}{Dt} + o(1). \tag{6.8}$$

Equation (6.3) for the flame speed can be now rewritten as

$$S_f = 1 - \delta\alpha\kappa. \tag{6.9}$$

Some have attempted to interpret the terms $\nabla^2 f$ and $\nabla . \mathbf{v}(0^-, y, z, t)$ in (6.2) as flame curvature and flame stretch respectively. Here we show that in fact both terms, when added to $N^{-1}\,DN/Dt$, which did not appear in other analyses, constitute the flame stretch κ. We observe that the deviation of flame speed from the flame speed of a plane adiabatic flame (i.e. $S_f = 1$) is directly proportional to flame stretch κ. Lewis & von Elbe (1967, p. 227) presented intuitive arguments indicating that positive stretch ($\kappa > 0$) implies a slower flame speed. Their argument is based on the idea that positive stretch lowers the rate at which heat and active species are transferred from the reaction zone to the unburned gases. Our results, however, indicate that flame stretch is also related to the mobility of the deficient reactant towards the flame. More precisely, a positive stretch ($\kappa > 0$) is associated with a decrease in the *local* flame speed only if

$$l > \frac{-2(1+q)\ln(1+q)}{q\int_{-\infty}^{0} \ln(1+qe^x)\,dx} \equiv -l^*, \tag{6.10}$$

i.e. if $Le > Le^*$, where $Le^* = 1 - \epsilon l^* < 1$.

Finally, we observe that our evolution equation (6.1) for the flame shape f, when specialized to the case where both f and $\mathbf{v}$ are $O(\delta)$, reduces to that obtained by Clavin & Williams (1982). Their correction term can be considered to be a linear version of the correction term in (6.1), since it is obtained from ours by setting $N = 1$. Then the term $N^{-1}\,DN/Dt$ vanishes and the term $N^{-1}\nabla^2 f$ reduces to $\nabla^2 f$, which is their result.

6.2. *The jump conditions across the flame*

Equations (1.5)–(1.7), (5.33), (5.36) and (5.43) imply that the jump conditions across the flame front $\xi = 0$ can be written as

$$[\![\rho(\mathbf{V}\,.\,\mathbf{n}-\nu_n)]\!] = \delta \ln(1+q)\kappa, \tag{6.11}$$

$$[\![\mathbf{V}\times\mathbf{n}]\!] = \delta\sigma\left\{N\left[\!\!\left[\frac{\partial}{\partial\xi}(\mathbf{V}\times\mathbf{n})\right]\!\!\right]-q\frac{\nabla N\times\mathbf{n}}{N}\right\}, \tag{6.12}$$

$$[\![p+\rho(\mathbf{V}\,.\,\mathbf{n})(\mathbf{V}\,.\,\mathbf{n}-\nu_n)]\!] = \sigma\left\{\frac{1+q}{q}\ln(1+q)\left(N\left[\!\!\left[\frac{\partial p}{\partial\xi}\right]\!\!\right]+q\frac{\nabla^2 f}{N}-\frac{q\nabla f\,.\,\nabla N}{N^2}\right)\right.$$
$$\left.+q\left(\frac{\nabla^2 f}{N}-\frac{\nabla f\,.\,\nabla N}{N^2}\right)+\nu_n\ln(1+q)\kappa\right\}. \tag{6.13}$$

We note that the dependence on the parameters accounts for the heat release q, and for viscous effects Pr. The dependence on the Prandtl number is of special interest, since the influence of viscosity on flame propagation is not yet well understood (cf. Markstein 1964; Frankel & Sivashinsky 1982). As noted earlier, the leading terms of the jump conditions (6.11)–(6.13) are identical with (1.5)–(1.7), namely with the Landau–Darrieus model. The $O(\delta)$ terms then provide corrections accounting for the interaction of fluid flow with species diffusion, heat conduction and chemical reactions, which were absent in the Landau–Darrieus model. If we again specialize the jump conditions to the case where both f and $\mathbf{v}$ are $O(\delta)$, they reduce to those obtained by Pelce & Clavin (1982).

We have thus derived a model for a flame in a general fluid flow. It consists of (4.9) and (4.10), the jump conditions (6.11)–(6.13) and the evolution equation (6.1).

6.3. *Vorticity production in the flame*

An aspect of gasdynamics associated with combustion, which has received some attention in the past, is the vorticity produced by the flame front. It is known that if the flow ahead of a flame is irrotational, a curved flame will generate vorticity so that the burned gases are no longer irrotational (see e.g. Emmons 1958). Indeed, even if the fluid upstream is rotational, the vorticity vector will jump across the flame. In order to quantify this property, we note that in the moving coordinates, the vorticity vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \nabla\times(u\mathbf{i}+\mathbf{v})+(\mathbf{i}-\nabla f)\times\frac{\partial\mathbf{v}}{\partial\xi}+(\mathbf{i}\times\nabla f)\frac{\partial u}{\partial\xi}. \tag{6.14}$$

Using (6.10) and (6.11), we find that to leading order

$$[\![\boldsymbol{\omega}]\!] = q\nabla\times\mathbf{n}+N\left[\!\!\left[\mathbf{n}\times\frac{\partial\mathbf{v}}{\partial\xi}\right]\!\!\right]+(\mathbf{i}\times\nabla f)\left[\!\!\left[\frac{\partial u}{\partial\xi}\right]\!\!\right]+o(1), \tag{6.15}$$

where $\mathbf{n}$ is a unit vector normal to the flame front. Therefore

$$[\![\omega\,.\,\mathbf{n}]\!] = o(1), \tag{6.16}$$

confirming that to leading order the normal component of the vorticity is continuous at the flame front, in agreement with the discussion in Emmons (1958, p. 610). Emmons' results follow from the assumption that the velocity components tangent to the flame surface are continuous at the flame. According to (6.12) this is only true to leading order in δ. Thus the jump in the normal component of $\boldsymbol{\omega}$ is $o(1)$, so that (6.15) represents the $O(1)$ jump in the tangential components of the vorticity vector. The vorticity thus produced at the flame is then carried into the region $\xi > 0$.

It should be noted that the jumps in the velocity gradients appearing in (6.15) can be evaluated from (5.32) and (5.37). The resulting equation for the jump in the vorticity is thus expressed in terms of the gas velocities ahead of the front. This is apparently the first time that an expression for the vorticity production is given explicitly in terms of the velocity field of the unburned gases.

6.4. *The temperature of the burned gases*

We recall that the temperature of the burned gases varies only by $O(\epsilon)$ from the adiabatic flame temperature $1+q$, and that

$$T = 1+q+\epsilon h(\xi, y, z, t) \quad (\xi > 0). \tag{6.17}$$

Expanding h as in (4.1) we found that H_0 must satisfy (4.3), implying that H_0 does not vary along particle paths. According to (5.13), the leading term in the expansion of h inside the flame zone is seen to vanish as $\zeta \rightarrow \infty$. Thus matching implies that $H_0(0, y, z, t) = 0$, and therefore

$$H_0 \equiv 0 \quad (\xi > 0). \tag{6.18}$$

Substituting the expansions (4.1) into (3.8) and using (6.18) we obtain an equation for H_1 as

$$\frac{\partial H_1}{\partial t} + S_0 \frac{\partial H_1}{\partial \xi} + \mathbf{V}_0 . \nabla H_1 = 0 \quad (\xi > 0). \tag{6.19}$$

Again the convective derivative of H_1 vanishes, implying that the values of H_1 at the flame are carried along particle paths into the burned region. Evaluation of $H_1(0, y, z, t)$ follows by matching to the large-ζ behaviour of the $O(\delta)$ term of the expansion of h inside the flame zone. The latter, based on (5.25) and (5.26), provides

$$H_1(0, y, z, t) = -lI\left(\frac{\nabla^2 f}{N} + \nabla . \mathbf{v}(0^-, y, z, t) + \frac{1}{N}\frac{DN}{Dt}\right). \tag{6.20}$$

In order to determine the temperature distribution of the burned gases, (6.19) must be solved for $\xi > 0$, subject to the boundary condition (6.20) at $\xi = 0$. We observe that the non-uniform temperature distribution is an $O(\epsilon\delta)$ correction to the adiabatic flame temperature.

Finally, we note that the flame temperature T_f is given by

$$T_f = 1+q-\epsilon\delta lI\left(\frac{\nabla^2 f}{N} + \nabla . \mathbf{v}(0^-, y, z, t) + \frac{1}{N}\frac{DN}{Dt}\right). \tag{6.21}$$

The terms in parentheses on the right-hand side of (6.21) have been identified as the flame stretch κ, so that $T_f = 1+q-\epsilon\delta lI\kappa$. Thus a positive stretch is associated with a decrease in flame temperature if $l > 0$ $(Le > 1)$ and with an increase of flame temperature if $l < 0$ $(Le < 1)$. We recall that a positive stretch is associated with a decrease in the local flame speed if $Le > Le^*$ with $Le^* < 1$, and an increase in flame speed otherwise. Thus, at least for some range of Lewis numbers, specifically for $Le^* < Le < 1$, an increase in flame temperature is not associated with an increase in flame speed.

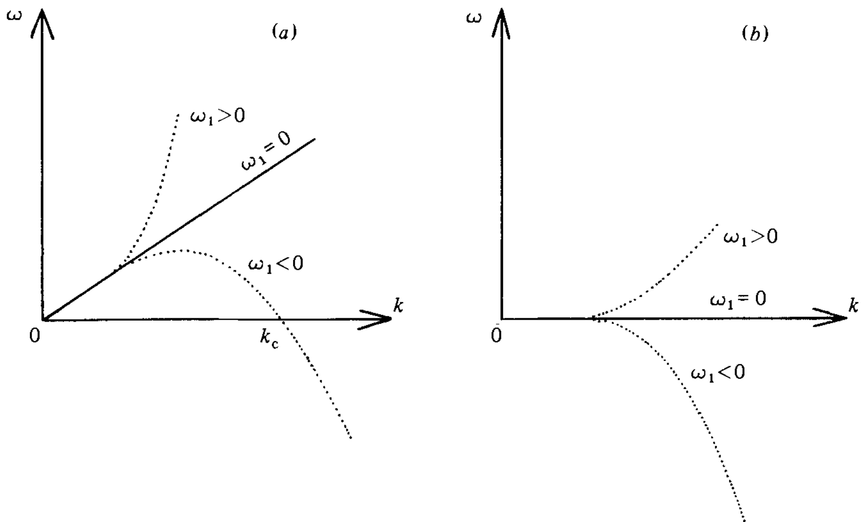

FIGURE 4. Growth rate of a disturbance versus its wavenumber according to results based on the model derived in this paper that takes account of the flame structure: (*a*) $q = O(1)$; (*b*) $q \ll 1$.

6.5. *The stability of plane flames*

We have employed the model derived above to study the stability of plane flames. Here we only summarize the results. Choosing the characteristic length L as the wavelength of the disturbance, the resulting dispersion relation may be expressed as a series in powers of the wavenumber k. We show that

$$\omega = \omega_0 k + \omega_1 k^2 + O(k^3), \tag{6.22}$$

with

$$\omega_0 = \frac{1+q}{2+q}\left\{\left[1+\frac{(2+q)q}{1+q}\right]^{\frac{1}{2}} - 1\right\}$$

$$\omega_1 = -\frac{(1+q)}{2}\left\{\frac{lI(1+\omega_0)(1+q+\omega_0)+q+[(1+q)/q]\ln(1+q)(2(1+\omega_0)+q)}{(1+q)+(2+q)\omega_0}\right\},$$

where $\omega_0 > 0$ was derived by Landau (1944) and Darrieus (1945), and ω_1 is a new correction term to their results, which depends on the parameters l and q. An expression similar to (6.22) was previously derived by Frankel & Sivashinsky (1982) and Pelce & Clavin (1982). We note that if $\omega_1 = 0$, plane flames are indeed unstable for all $k > 0$, as envisaged by Landau & Darrieus (this is often called the hydrodynamical instability). The correction $O(k^2)$ term in (6.22) has a stabilizing effect for $k > k_c$ (see figure 4) if $\omega_0 < 0$, and a destabilizing effect otherwise. Whether ω_1 is negative or not depends on the Lewis number expressed by l. In particular we note that for $q \ll 1$, $\omega_0 \sim \frac{1}{2}q$, so that the hydrodynamic instability disappears (figure 4*b*). Then plane flames are shown to be stable for $lq > -2$, a condition first derived by Sivashinsky (1976) and referred to as the diffusional–thermal stability.

This research was supported in part by D.O.E. Grant DE-ACO2-ERO4650 and by A.R.O. Grant DAAG-79-C-0183.

REFERENCES

BARENBLATT, G. I., ZELDOVICH, Y. B. & ISTRATOV, A. G. 1962 On diffusional thermal instability of laminar flame. *Prikl. Mekh. Tekh. Fiz.* **2**, 21.

BUCKMASTER, J. D. 1979 The quenching of two dimensional premixed flames. *Acta Astronautica* **6**, 741.

CLAVIN, P. & WILLIAMS, F. A. 1982 Effects of molecular diffusion and of thermal expansion on the structure and dynamics of premixed flames in turbulent flows of large scales and low intensity. *J. Fluid Mech.* **116**, 251.

DARRIEUS, G. 1945 Propagation d'un front de flamme. Presented at *Le congres de Mecanique Appliquee* (unpublished).

ECKHAUS, W. 1961 Theory of flame-front stability. *J. Fluid Mech.* **10**, 80.

EMMONS, H. W. 1958 Flow discontinuities associated with combustion. In *Fundamentals of Gas Dynamics* (ed. H. W. Emmons), p. 584. Princeton University Press.

FRANKEL, M. L. & SIVASHINSKY, G. I. 1982 The effect of viscosity on hydro-dynamic stability of a plane flame front. Submitted for publication.

HARTEN, A. VAN & MATKOWSKY, B. J. 1982 A new model in flame theory. *SIAM J. Appl. Math.* (to appear).

KARLOVITZ, B., DENNISTON, D. W., KNAPSCHAEFER, D. H. & WELLS, F. E. 1953 Studies on turbulent flames. in *Proc. 4th Int. Symp. on Combustion*, p. 613. William & Wilkins.

LANDAU, L. D. 1944 On the theory of slow combustion. *Acta Physicochimica URSS* **19**, 77.

LEWIS, B. & ELBE, VON G. 1967 *Combustion Flames and Explosion of Gases*, 2nd edn. Academic.

MARKSTEIN, G. H. 1951 Experimental and theoretical studies of flame front stability. *J. Aero. Sci.* **18**, 199.

MARKSTEIN, G. H. 1964 *Nonsteady Flame Propagation*. Pergamon.

MATKOWSKY, B. J. & SIVASHINSKY, G. I. 1979 An asymptotic derivation of two models in flame theory associated with the constant density approximation. *SIAM J. Appl. Math.* **37**, 686.

PELCE, P. & CLAVIN, R. 1982 Influence of hydrodynamics and diffusion upon the stability limits of laminar premixed flames. *J. Fluid Mech.* **124**, 219.

SIVASHINSKY, G. I. 1976 On a distorted flame front as a hydrodynamic discontinuity. *Acta Astronautica* **3**, 889.